# A NOTE ON THE SECOND MOMENT OF AUTOMORPHIC $L$-FUNCTIONS

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Abstract. We obtain the formula for the twisted harmonic second moment of the $L$-functions associated with primitive Hecke eigenforms of weight 2. A consequence of our mean value theorem is reminiscent of recent results of Conrey and Young on the reciprocity formula for the twisted second moment of Dirichlet $L$-functions.

## 1. Introduction

In this paper, we study the twisted second moment of the family of $L$-functions arising from $\mathcal{S}_2^*(q)$, the set of primitive Hecke eigenforms of weight 2, lever $q$ ($q$ prime). For $f(z)\in\mathcal{S}_2^*(q)$, $f$ has a Fourier expansion

$$f(z)=\sum_{n=1}^{\infty}n^{1/2}\lambda_f(n)e(nz),$$

where the normalization is such that $\lambda_f(1)=1$. The $L$-function associated to $f$ has an Euler product

$$L(f,s)=\sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s}=\bigg(1-\frac{\lambda_f(q)}{q^s}\bigg)^{-1}\prod_{\substack{h\ \textrm{prime}\\ h\neq q}}\bigg(1-\frac{\lambda_f(h)}{h^s}+\frac{1}{h^{2s}}\bigg)^{-1}.$$

The series is absolutely convergent when $\Re s>1$, and admits analytic continuation to all of $\mathbb{C}$. The functional equation for $L(f,s)$ is

$$\Lambda(f,s):=\bigg(\frac{\sqrt{q}}{2\pi}\bigg)^{s}\Gamma(s+\tfrac{1}{2})L(f,s)=\varepsilon_f\Lambda(f,1-s),$$

where $\varepsilon_f=-q^{1/2}\lambda_f(q)=\pm1$. We define the harmonic average as

$$\sum_{f}{}^{h}A_f:=\sum_{f\in\mathcal{S}_2^*(q)}\frac{A_f}{4\pi(f,f)},$$

where $(f,g)$ is the Petersson inner product on the space $\Gamma_0(q)\backslash\mathbb{H}$.

We are interested in the twisted second moment of this family of $L$-functions. We define

$$S(p,q)=\sum_{f\in\mathcal{S}_2^*(q)}{}^{h}L(f,\tfrac{1}{2})^2\lambda_f(p).$$

Our main theorem is

**Theorem 1.** *Suppose $q$ is prime and $0<p\leq Cq$, for some fixed $C<1$. Then we have*

$$S(p,q)=\frac{d(p)}{\sqrt{p}}\log\frac{q}{4\pi^2p}+O(p^{1/2}q^{-1+\varepsilon}).$$

**Remark 1.** The twisted harmonic fourth moment has been considered by Kowalski, Michel and VanderKam [**6**], where they gave an asymptotic formula for the fourth power mean value provided that $p \ll q^{1/9-\varepsilon}$.

**Remark 2.** In a similar setting, Iwaniec and Sarnak [**3**] have given the exact formula for the twisted second moment of the automorphic $L$-functions arising from $\mathcal{H}_k(1)$, the set of newforms in $\mathcal{S}_k(1)$, where $\mathcal{S}_k(1)$ is the linear space of holomorphic cusp forms of weight $k$. Precisely, they showed that for $k > 2$, $k \equiv 0(\text{mod } 2)$, and for any $m \geq 1$, we have

$$\frac{12}{k-1}\sum_{f\in\mathcal{H}_k(1)} w_f L(f,\tfrac12)^2\lambda_f(m) = 2(1+i^k)\frac{d(m)}{\sqrt{m}}\bigg(\sum_{0<l\leq k/2}\frac{1}{l} - \log 2\pi\sqrt{m}\bigg)$$
$$-\frac{2\pi i^k}{\sqrt{m}}\sum_{h\neq m} d(h)d(h-m)p_k\Big(\frac{h}{m}\Big) + \frac{2\pi i^k}{\sqrt{m}}\sum_{h} d(h)d(h+m)q_k\Big(\frac{h}{m}\Big),$$

where $p_k(x)$ and $q_k(x)$ are Hankel transforms of Bessel functions

$$p_k(x) = \int_0^\infty Y_0(y\sqrt{x})J_{k-1}(y)dy, \text{ and } q_k(x) = \frac{2}{\pi}\int_0^\infty K_0(y\sqrt{x})J_{k-1}(y)dy.$$

Here the weight $w_f = \zeta(2)L(\text{sym}^2(f),1)^{-1}$, where the symmetric square $L$-function $L(\text{sym}^2(f),s)$ corresponding to $f$ is defined by

$$L(\text{sym}^2(f),s) = \zeta(2s)\sum_{n=1}^\infty\frac{\lambda_f(n^2)}{n^s}.$$

In the context of Dirichlet $L$-functions, consider

$$M(p,q) = \frac{1}{\varphi^*(q)}\sum_{\chi(\text{mod } q)}^{*} |L(\tfrac12,\chi)|^2\chi(p),$$

where $\sum^*$ denotes summation over all primitive characters $\chi(\text{mod } q)$, and $\varphi^*(q)$ is the number of primitive characters. This is the twisted second moment of Dirichlet $L$-functions. In a recent paper, Conrey [**1**] proved that there is a kind of reciprocity formula relating $M(p,q)$ and $M(-q,p)$ when $p$ and $q$ are distinct prime integers. Precisely, Conrey showed that

$$M(p,q) = \frac{\sqrt{p}}{\sqrt{q}}M(-q,p) + \frac{1}{\sqrt{p}}\bigg(\log\frac{q}{p} + A\bigg) + \frac{B}{2\sqrt{q}} + O\bigg(\frac{p}{q} + \frac{\log q}{q} + \frac{\log pq}{\sqrt{pq}}\bigg),$$

where $A$ and $B$ are some explicit constants. This provides an asymptotic formula for $M(p,q) - \sqrt{p/q}M(-q,p)$ under the condition that $p \ll q^{2/3-\varepsilon}$. The error term above was improved by Young [**7**] so that the asymptotic formula holds for $p \ll q^{1-\varepsilon}$.

We now take $p$ to be prime and, similarly as before, $S(q,p)$ is defined as the harmonic second moment, twisted by $\lambda_g(q)$, of the family of $L$-functions arising from $g(z) \in \mathcal{S}_2^*(p)$. We note that as $q$ is prime, the Ramanujan bound $|\lambda_f(n)| \leq d(n)$ [**2**] yields

$$S(q,p) \ll \sum_{g\in\mathcal{S}_2^*(p)}^{h} L(g,\tfrac12)^2 \ll \log p.$$

Thus as a trivial consequence of Theorem 1, for $p < q$ we have

$$S(p,q) - \sqrt{p/q}S(q,p) = \frac{2}{\sqrt{p}}\log\frac{q}{4\pi^2 p} + O(p^{1/2+\varepsilon}q^{-1/2}).$$

This leads to an asymptotic formula for $S(p,q) - \sqrt{p/q}S(q,p)$, at least for $p$ as large as $q^{1/2-\varepsilon}$. The results in the Dirichlet $L$-functions case [**1**,**7**] suggest that the asymptotic formula should hold for $p \ll q^{\theta}$, for any $\theta < 1$. However, our technique fails to extend the range to any power $\theta > 1/2$. For that purpose, we need more refined estimates for the off-diagonal terms of $S(p,q)$ and $S(q,p)$. The intricate calculations seem to suggest that there is a large cancellation between these two expressions. The nature of this is not well-understood.

## 2. Preliminary lemmas

We require some lemmas. We begin with Hecke's formula for primitive forms.

**Lemma 1.** *For $m, n \geq 1$,*

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n)\\(d,q)=1}} \lambda_f\left(\frac{mn}{d^2}\right).$$

The next lemma is a particular case of Petersson's trace formula.

**Lemma 2.** *For $m, n \geq 1$, we have*

$$\sum_{f\in\mathcal{S}_2^*(q)}{}^{h}\, \lambda_f(m)\lambda_f(n) = \delta_{m,n} - J_q(m,n),$$

*where $\delta_{m,n}$ is the Kronecker symbol and*

$$J_q(m,n) = 2\pi\sum_{c=1}^{\infty}\frac{S(m,n;cq)}{cq}J_1\left(\frac{4\pi\sqrt{mn}}{cq}\right).$$

*Here $J_1(x)$ is the Bessel function of order 1, and $S(m,n;c)$ is the Kloosterman sum*

$$S(m,n;c) = \sum_{a(\mathrm{mod}\ c)}{}^{*}\, e\left(\frac{ma+n\overline{a}}{c}\right).$$

*Moreover we have*

$$J_q(m,n) \ll (m,n,q)^{1/2}(mn)^{1/2+\varepsilon}q^{-3/2}.$$

The above estimate follows easily from the bound $J_1(x) \ll x$ and Weil's bound on Kloosterman sums.

We mention a result of Jutila [**4**] (cf. Theorem 1.7), which is an extension of the Voronoi summation formula.

**Lemma 3.** *Let $f : \mathbb{R}^+ \to \mathbb{C}$ be a $C^\infty$ function which vanishes in the neighbourhood of $0$ and is rapidly decreasing at infinity. Then for $c \geq 1$ and $(a,c) = 1$,*

$$\begin{aligned}
c\sum_{m=1}^{\infty} d(m)e\left(\frac{am}{c}\right)f(m) &= 2\int_0^\infty (\log\frac{\sqrt{x}}{c}+\gamma)f(x)dx \\
&\quad -2\pi\sum_{m=1}^{\infty}d(m)e\left(\frac{-\overline{a}m}{c}\right)\int_0^\infty Y_0\left(\frac{4\pi\sqrt{mx}}{c}\right)f(x)dx \\
&\quad +4\sum_{m=1}^{\infty}d(m)e\left(\frac{\overline{a}m}{c}\right)\int_0^\infty K_0\left(\frac{4\pi\sqrt{mx}}{c}\right)f(x)dx.
\end{aligned}$$

The next lemma concerns the approximate functional equation for $L$-functions.

**Lemma 4.** *Let $G(s)$ be an even entire function satisfying $G(0)=1$ and $G$ has a double zero at each $s\in\mathbb{Z}$. Furthermore let assume that $G(s)\ll_{A,B}(1+|s|)^{-A}$ for any $A>0$ in any strip $-B\le\Re s\le B$. Then for $f\in\mathcal{S}_2^*(q)$,*

$$L(f,\tfrac12)^2=2\sum_{n=1}^{\infty}\frac{d(n)\lambda_f(n)}{\sqrt{n}}W_q\Big(\frac{4\pi^2n}{q}\Big),$$

*where*

$$W_q(x)=\frac{1}{2\pi i}\int_{(1)}G(s)\Gamma(s+1)^2\zeta_q(2s+1)x^{-s}\frac{ds}{s}.$$

*Here $\zeta_q(s)$ is defined by*

$$\zeta_q(s)=\sum_{\substack{n=1\\(n,q)=1}}^{\infty}n^{-s}\qquad(\sigma>1).$$

*Proof.* From Lemma 1 we first note that

$$L(f,s)^2=\zeta_q(2s)\sum_{n=1}^{\infty}\frac{d(n)\lambda_f(n)}{n^s}\qquad(\sigma>1).$$

Consider

$$A(f):=\frac{1}{2\pi i}\int_{(1)}\frac{G(s)\Lambda(f,s+\frac12)^2}{\frac{\sqrt{q}}{2\pi}}\frac{ds}{s}.$$

Moving the line of integration to $\Re s=-1$, and applying Cauchy's theorem and the functional equation, we derive that $A(f)=L(f,\frac12)^2-A(f)$. Expanding $\Lambda(f,s+\frac12)^2$ in a Dirichlet series and integrating termwise we obtain the lemma. □

For our purpose, $W_q$ is basically a "cut-off" function. Indeed, we have the following.

**Lemma 5.** *The function $W_q$ satisfies*

$$W_q^{(j)}(x)\ll_{j,N}x^{-N}\ \text{for } x\ge1 \text{ and all } j,N\ge0,$$

$$x^iW_q^{(j)}(x)\ll_{i,j}|\log x|\ \text{for } 0<x<1 \text{ and all } i\ge j\ge0,$$

*and*

$$W_q(x)=-\Big(1-\frac1q\Big)\frac{\log x}{2}+\frac{\log q}{q}+O_N(x^N)\ \text{for } 0<x<1 \text{ and all } N\ge0.$$

*The implicit constants are independent of $q$.*

*Proof.* The first estimate is a direct consequence of Stirling's formula after differentiating under the integral sign and shifting the line of integration to $\Re s=N$. The only difference in the other two estimates is that one has to move the line of integration to $\Re s=-N$. □

## 3. Proof of Theorem 1

Our argument in this section follows closely [**5**]. From Lemma 4 and Lemma 2 we obtain

$$S(p,q)=2\frac{d(p)}{\sqrt{p}}W_q\left(\frac{4\pi^2p}{q}\right)-2R(p,q),$$

where

$$R(p,q)=\sum_{n=1}^{\infty}\frac{d(n)}{\sqrt{n}}J_q(n,p)W_q\left(\frac{4\pi^2n}{q}\right).$$

Using Lemma 5, the first term is

$$\frac{d(p)}{\sqrt{p}}\log\frac{q}{4\pi^2p}+O(p^{-1/2}q^{-1+\varepsilon}+p^{1/2+\varepsilon}q^{-1}).$$

Thus, we are left to consider $R(p,q)$. We have

$$R(p,q)=2\pi\sum_{n=1}^{\infty}\frac{d(n)}{\sqrt{n}}\sum_{c=1}^{\infty}\frac{S(n,p;cq)}{cq}J_1\left(\frac{4\pi\sqrt{np}}{cq}\right)W_q\left(\frac{4\pi^2n}{q}\right).$$

Using Weil's bound for Kloosterman sums and $J_1(x)\ll x$, the contribution from the terms $c\geq q$ is

$$\ll p^{1/2}q^{-3/2}\sum_{n=1}^{\infty}(n,p)^{1/2}d(n)\left|W_q\left(\frac{4\pi^2n}{q}\right)\right|\sum_{c\geq q}\frac{d(c)}{c^{3/2}}\ll p^{1/2}q^{-1+\varepsilon}.$$

Thus we need to study

$$\frac{2\pi}{q}\sum_{c<q}\frac{1}{c}\sideset{}{^*}\sum_{a(\mathrm{mod}\ cq)}e\left(\frac{\overline{a}p}{cq}\right)\sum_{n=1}^{\infty}d(n)e\left(\frac{an}{cq}\right)\frac{J_1(\frac{4\pi\sqrt{np}}{cq})W_q(\frac{4\pi^2n}{q})}{\sqrt{n}}.$$

We fix a $C^\infty$ function $\xi:\mathbb{R}^+\to[0,1]$, which satisfies $\xi(x)=0$ for $0\leq x\leq 1/2$ and $\xi(x)=1$ for $x\geq 1$, and attach the weight $\xi(n)$ to the innermost sum. Using Lemma 3, this is equal to

$$\frac{4\pi}{q^2}\sum_{c<q}\frac{1}{c^2}S(0,p;cq)\int_0^\infty(\log\frac{\sqrt{t}}{cq}+\gamma)J_1\left(\frac{4\pi\sqrt{tp}}{cq}\right)W_q\left(\frac{4\pi^2t}{q}\right)\xi(t)\frac{dt}{\sqrt{t}}-Y+K,$$

where

$$\begin{aligned}Y &= \frac{4\pi^2}{q^2}\sum_{c<q}\frac{1}{c^2}\sum_{n=1}^{\infty}d(n)S(0,p-n;cq)\\ &\qquad\int_0^\infty Y_0\left(\frac{4\pi\sqrt{nt}}{cq}\right)J_1\left(\frac{4\pi\sqrt{tp}}{cq}\right)W_q\left(\frac{4\pi^2t}{q}\right)\xi(t)\frac{dt}{\sqrt{t}},\end{aligned}\tag{1}$$

and

$$\begin{aligned}K &= \frac{8\pi}{q^2}\sum_{c<q}\frac{1}{c^2}\sum_{n=1}^{\infty}d(n)S(0,p+n;cq)\\ &\qquad\int_0^\infty K_0\left(\frac{4\pi\sqrt{nt}}{cq}\right)J_1\left(\frac{4\pi\sqrt{tp}}{cq}\right)W_q\left(\frac{4\pi^2t}{q}\right)\xi(t)\frac{dt}{\sqrt{t}}.\end{aligned}\tag{2}$$

We will deal with $Y$ and $K$ in the next three lemmas. For the first sum, since $S(0,p;cq)=\mu(q)S(0,p\overline{q};c)$ and $J_1(x)\ll x$, this is

$$\ll p^{1/2}q^{-3}\sum_{c<q}\frac{1}{c^2}\int_{1/2}^{\infty}\Big|W_q\Big(\frac{4\pi^2t}{q}\Big)\Big|(\log tcq)dt\ll p^{1/2}q^{-2+\varepsilon}.$$

**Lemma 6.** *For $K$ defined as in (2), we have*

$$K\ll p^{1/2}q^{\varepsilon}(q-p)^{-1+\varepsilon}.$$

*And hence $K\ll p^{1/2}q^{-1+\varepsilon}$, given that $p\leq Cq$ for some fixed $C<1$.*

**Remark 3.** This is the only place where the condition $p\leq Cq$ for some constant $C<1$ is used.

*Proof.* The integral involving $K_0$, using $K_0(y)\ll y^{-1/2}e^{-y}$, is

$$\begin{aligned}&\int_0^{\infty}K_0\Big(\frac{4\pi\sqrt{nt}}{cq}\Big)J_1\Big(\frac{4\pi\sqrt{tp}}{cq}\Big)W_q\Big(\frac{4\pi^2t}{q}\Big)\xi(t)\frac{dt}{\sqrt{t}}\\ =\ &\frac{cq}{2\pi\sqrt{n}}\int_0^{\infty}K_0(y)J_1\Big(\sqrt{\frac{p}{n}}y\Big)W_q\Big(\frac{c^2qy^2}{4n}\Big)\xi\Big(\frac{c^2q^2y^2}{16\pi^2n}\Big)dy\\ \ll\ &\frac{cp^{1/2}q^{1+\varepsilon}}{n}\int_{\sqrt{n}/cq}^{\infty}y^{1/2}e^{-y}dy\ll\frac{cp^{1/2}q^{1+\varepsilon}}{n}e^{-\sqrt{n}/2cq}.\end{aligned}$$

Thus, as $S(0,p+n;cq)=S(0,(p+n)\overline{q};c)S(0,p+n;q)$ and $|S(0,(p+n)\overline{q};c)|\leq\sum_{l|(p+n,c)}l$,

$$\begin{aligned}K\ &\ll\ p^{1/2}q^{-1+\varepsilon}\sum_{n=1}^{\infty}\frac{d(n)}{n}e^{-\sqrt{n}/2q^2}|S(0,p+n;q)|\sum_{c<q}\frac{\sum_{l|(p+n,c)}l}{c}\\ &\ll\ p^{1/2}q^{-1+\varepsilon}\sum_{n=1}^{\infty}\frac{d(n)d(p+n)}{n}e^{-\sqrt{n}/2q^2}|S(0,p+n;q)|.\end{aligned}$$

We break the sum over $n$ according to whether $q|(p+n)$ or $q\nmid(p+n)$. The contribution of the latter is $O(p^{1/2}q^{-1+\varepsilon})$. That of the former is

$$\ll p^{1/2}q^{\varepsilon}\sum_{l=1}^{\infty}\frac{d(l)d(ql-p)}{ql-p}e^{-\sqrt{ql-p}/2q^2}\ll p^{1/2}q^{\varepsilon}(q-p)^{-1+\varepsilon}+p^{1/2}q^{-1+\varepsilon}.$$

The lemma follows. □

The case of $Y$ is more complicated as $Y_0$ is an oscillating function. For that we need the following standard lemma (for example, see [**5**]).

**Lemma 7.** *Let $v\geq 0$ and $J$ be a positive integer. If $f$ is a compactly supported $C^{\infty}$ function on $[Y,2Y]$, and there exists $\beta>0$ such that*

$$y^jf^{(j)}(y)\ll_j(1+\beta Y)^j$$

*for $0\leq j\leq J$, then for any $\alpha>1$, we have*

$$\int_0^{\infty}Y_v(\alpha y)f(y)dy\ll\Big(\frac{1+\beta Y}{1+\alpha Y}\Big)^JY.$$

**Lemma 8.** *For $Y$ defined as in (1), we have*

$$Y\ll p^{1/2}q^{-1+\varepsilon}.$$

*Proof.* We have

$$Y = \frac{4\pi^2}{q^2} \sum_{c<q} \frac{1}{c^2} \sum_{n=1}^{\infty} d(n) S(0, p-n; cq) y(n), \tag{3}$$

where

$$y(n) = \int_0^\infty Y_0\left(\frac{4\pi\sqrt{nt}}{cq}\right) J_1\left(\frac{4\pi\sqrt{tp}}{cq}\right) W_q\left(\frac{4\pi^2 t}{q}\right) \xi(t) \frac{dt}{\sqrt{t}}. \tag{4}$$

We make a smooth dyadic partition of unity that $\xi = \sum_k \xi_k$, where each $\xi_k$ is a compactly supported $C^\infty$ function on the dyadic interval $[X_k, 2X_k]$. Moreover, $\xi_k$ satisfies $x^j \xi_k^{(j)}(x) \ll 1$, for all $j \geq 0$. We work on each $\xi_k$ individually, but we write $\xi$ instead of $\xi_k$ and, accordingly, $X$ rather than $X_k$.

By the change of variable $x := 2\sqrt{t}/cq$, we have

$$y(n) = cq \int_0^\infty Y_0(2\pi\sqrt{n}x) J_1(2\pi\sqrt{p}x) W_q(\pi^2 c^2 q x^2) \xi\left(\frac{c^2 q^2 x^2}{4}\right) dx.$$

We define

$$f(x) := J_1(2\pi\sqrt{p}x) W_q(\pi^2 c^2 q x^2) \xi\left(\frac{c^2 q^2 x^2}{4}\right).$$

This is a $C^\infty$ function compactly supported on $[\rho, 2\rho]$, where $\rho = 2\sqrt{X}/cq$.

We first treat the case $1/2 \leq X \leq q$. We note that this involves $O(\log q)$ dyadic intervals. From Lemma 5 we have $x^j W^{(j)}(x) \ll_j \log q$ for $1/q \ll x \ll 1$. This, together with the recurrence relation $(x^v J_v(x))' = x^v J_{v-1}(x)$, gives

$$x^j f^{(j)}(x) \ll_j (1 + \sqrt{p}x)^j \log q. \tag{5}$$

We are in a position to apply Lemma 7 to $f$ with $\alpha = 2\pi\sqrt{n}$, $\beta = \sqrt{p}$ and $Y = \rho = 2\sqrt{X}/cq$. The lemma yields, for any positive integer $J$,

$$y(n) \ll cq\rho \left(\frac{1 + \sqrt{p}\rho}{1 + \sqrt{n}\rho}\right)^J \log q. \tag{6}$$

Later, we will break the sum over $n$ in (3) in the following way

$$\sum_{n \geq 1} = \sum_{n \leq \rho^{-\kappa}} + \sum_{n > \rho^{-\kappa}},$$

where $\kappa > 2$ will be chosen later. The estimate (6) will be used for $n > \rho^{-\kappa}$. We need another estimate for the range $n \leq \rho^{-\kappa}$. For this we go back to (4), using $Y_0(x) \ll 1 + |\log x|$ and $J_1(x) \ll x$, to derive

$$y(n) \ll \frac{\sqrt{p}X}{cq} (\log q)^2. \tag{7}$$

We denote by $Y_1$ and $Y_2$ the corresponding splitted sums ($Y = Y_1 + Y_2$). For the first sum, using (7), we have

$$\begin{aligned}
Y_1 &= \frac{4\pi^2}{q^2}\sum_{c<q}\frac{1}{c^2}\sum_{n\le\rho^{-\kappa}} d(n)S(0,p-n;cq)y(n)\\
&\ll p^{1/2}Xq^{-3+\varepsilon}\sum_{n\le\rho^{-\kappa}} d(n)|S(0,p-n;q)|\sum_{c<q}\frac{1}{c^3}\sum_{l|(p-n,c)} l\\
&\ll p^{1/2}Xq^{-3+\varepsilon}\sum_{n\le(\frac{q^2}{2\sqrt{X}})^\kappa} d(n)|S(0,p-n;q)|\sum_{\frac{2\sqrt{X}}{q}n^{1/\kappa}\le c<q}\ \sum_{l|(p-n,c)}\frac{l}{c^3}\\
&\ll p^{1/2}q^{-1+\varepsilon}\sum_{n\le(\frac{q^2}{2\sqrt{X}})^\kappa}\frac{d(n)d(p-n)}{n^{2/\kappa}}|S(0,p-n;q)|\\
&\ll p^{1/2}q^{2\kappa-5+\varepsilon}.
\end{aligned}\tag{8}$$

For the second sum, we note that $\sqrt{p}\rho \ll 1$ in this range. Using (6), we have

$$y(n) \ll \sqrt{X}(\log q)n^{-J/2}\rho^{-J}.$$

Similarly to above, we deduce that

$$\begin{aligned}
Y_2 &= \frac{4\pi^2}{q^2}\sum_{c<q}\frac{1}{c^2}\sum_{n>\rho^{-\kappa}} d(n)S(0,p-n;cq)y(n)\\
&\ll \sqrt{X}q^{-2+\varepsilon}\sum_{n>\rho^{-\kappa}}\frac{d(n)}{n^{J/2}}|S(0,p-n;q)|\sum_{c<q}\frac{1}{c^2}\sum_{l|(p-n,c)} l\rho^{-J}\\
&\ll X^{-(J-1)/2}q^{J-2}\sum_{n}\frac{d(n)}{n^{J/2}}|S(0,p-n;q)|\sum_{l|p-n} l^{J-1}\sum_{c\le\frac{2\sqrt{X}}{ql}n^{1/\kappa}} c^{J-2}\\
&\ll q^{-1+\varepsilon}\sum_{n}\frac{d(n)d(p-n)}{n^{J/2-(J-1)/\kappa}}|S(0,p-n;q)|.
\end{aligned}\tag{9}$$

To this end, we choose $\kappa = 2 + \varepsilon/2$ and $J$ large enough so that $J/2 - (J-1)/\kappa > 1$. We hence obtain $Y_1 \ll p^{1/2}q^{-1+\varepsilon}$ and, since the sum over $n$ in (9) converges, $Y_2 \ll q^{-1+\varepsilon}$.

For $X > q$, similarly to (5), using the bound $x^jW^{(j)}(x) \ll_j x^{-2}$, we have

$$x^jf^{(j)}(x) \ll_j (1+\sqrt{p}x)^jq^{-2}(cx)^{-4}.$$

Lemma 7 then gives

$$y(n) \ll cq\rho\left(\frac{1+\sqrt{p}\rho}{1+\sqrt{n}\rho}\right)^J\frac{q^2}{X^2}.$$

For the range $n \le \rho^{-\kappa}$, a better bound than (7) in this case is

$$y(n) \ll \frac{\sqrt{p}X}{cq}(\log q)\frac{q^2}{X^2}.$$

Since $q^2/X^2 \ll 1$, all the previous estimates remain valid. The ony place where this is not the case is the sum over $n \ll (q^2/2\sqrt{X})^\kappa$ in (8). However, this sum is void for $X > q^4/4$ and the former estimate still works in the larger interval $X \le q^4/4$. Also, the

quantity saved $q^2/X^2$ is sufficient to allow the sum over the dyadic values of $X$ involved to converge. The lemma follows. $\square$

The proof of the theorem is complete.

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